# ON THE DISTRIBUTION OF COMPLEX ROOTS OF RANDOM POLYNOMIALS WITH HEAVY-TAILED COEFFICIENTS

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Abstract. Consider a random polynomial $G_n(z) = \xi_n z^n + \dots + \xi_1 z + \xi_0$ with i.i.d. complex-valued coefficients. Suppose that the distribution of $\log(1 + \log(1 + |\xi_0|))$ has a slowly varying tail. Then the distribution of the complex roots of $G_n$ concentrates in probability, as $n \to \infty$, to two centered circles and is uniform in the argument as $n \to \infty$. The radii of the circles are $|\xi_0/\xi_\tau|^{1/\tau}$ and $|\xi_\tau/\xi_n|^{1/(n-\tau)}$, where $\xi_\tau$ denotes the coefficient with the maximum modulus.

*Key words and concepts:* roots of a random polynomial, roots concentration, heavy-tailed coefficients

## 1. Introduction

Consider the sequence of random polynomials

$$G_n(z) = \xi_n z^n + \xi_{n-1} z^{n-1} + \dots + \xi_1 z + \xi_0,$$

where $\xi_0, \xi_1, \dots, \xi_n, \dots$ are i.i.d. real- or complex-valued random variables. We would like to investigate the behaviour of the complex roots of $G_n$.

The first results in this questions are due to Hammersley [2]. He derived an explicit formula for the $r-$point correlation function ($1 \leqslant r \leqslant n$) of the roots of $G_n$ when the coefficients have an arbitrary joint distribution.

Shparo and Shur [9] showed that under quite general assumptions the roots of $G_n$ concentrate near the unit circle as $n$ tends to $\infty$ with asymptotically uniform distribution of the argument. More precisely, denote by $R_n(a, b)$ respectively $S_n(\alpha, \beta)$ the number of the roots of $G_n$ contained in the ring $\{z \in \mathbb{C} : a \leq |z| \leq b\}$ respectively the sector $\{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta\}$. For $\varepsilon > 0, m \in \mathbb{Z}_+$ consider the function

$$f(t) = \Bigg[\underbrace{\log^+ \log^+ \dots \log^+ t}_{m+1}\Bigg]^{1+\varepsilon} \cdot \prod_{k=1}^{m} \underbrace{\log^+ \log^+ \dots \log^+ t}_{k},$$

where $\log^+ s = \max(1, \log s)$. If for some $\varepsilon > 0, m \in \mathbb{Z}^+$

$$\mathbf{E} f(|\xi_0|) < \infty,$$

then for any $\delta \in (0, 1)$ and any $\alpha, \beta$ such that $-\pi \leqslant \alpha < \beta \leqslant \pi$

$$\frac{1}{n} R_n(1 - \delta, 1 + \delta) \xrightarrow{\mathbf{P}} 1, \quad n \to \infty,$$

$$\frac{1}{n} S_n(\alpha, \beta) \xrightarrow{\mathbf{P}} \frac{\beta - \alpha}{2\pi}, \quad n \to \infty.$$

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Ibragimov and Zaporozhets [4] improved this result as follows. They showed that

$$\mathbf{P}\left\{\frac{1}{n}R_n(1-\delta,1+\delta)\underset{n\to\infty}{\longrightarrow}1\right\}=1$$

holds for any $\delta\in(0,1)$ if and only if

$$\mathbf{E}\log(1+|\xi_0|)<\infty.$$

They also proved that for any $\alpha,\beta$ such that $-\pi\leqslant\alpha<\beta\leqslant\pi$

$$\mathbf{P}\left\{\frac{1}{n}S_n(\alpha,\beta)\underset{n\to\infty}{\longrightarrow}\frac{\beta-\alpha}{2\pi}\right\}=1$$

holds for *any* distribution of $\xi_0$.

Shepp and Vanderbei [8] considered real-valued standard Gaussian coefficients and proved that

$$\frac{1}{n}\mathbf{E}R_n(e^{-\delta/n},e^{\delta/n})\longrightarrow\frac{1+e^{-2\delta}}{1-e^{-2\delta}}-\frac{1}{\delta},\quad n\to\infty$$

for any $\delta>0$. Ibragimov and Zeitouni [3] extended this relation to the case of arbitrary i.i.d. coefficients from the domain of attraction of an $\alpha$-stable law:

$$\frac{1}{n}\mathbf{E}R_n(e^{-\delta/n},e^{\delta/n})\longrightarrow\frac{1+e^{-\alpha\delta}}{1-e^{-\alpha\delta}}-\frac{2}{\alpha\delta},\quad n\to\infty. \tag{1}$$

It is interesting to consider the limit case when $\alpha\to 0$. Then

$$\frac{1+e^{-\alpha\delta}}{1-e^{-\alpha\delta}}-\frac{2}{\alpha\delta}\longrightarrow 0$$

and a natural assumption for the coefficient distribution would be a slowly varying tail. In this case (1) becomes

$$\frac{1}{n}\mathbf{E}R_n(e^{-\delta/n},e^{\delta/n})\longrightarrow 0,\quad n\to\infty.$$

This result (in a slightly stronger form) is proved in Theorem 1.

In contrast to the concentration near the unit circumference, there exist random polynomials with quite a different asymptotic behavior of complex roots. Zaporozhets [10] constructed a random polynomial with i.i.d. coefficients such that in average $n/2+o(1)$ of the complex roots concentrate near the origin and the same number tends to infinity as $n\to\infty$ (moreover, the expected number of *real* roots of this polynomial is at most 9 for all $n$). Theorem 2 generalizes this result.

The paper is organized as follows. In Sect. 2 we formulate our results. In Sect. 3 we prove some auxiliary lemmas. The theorems are proved in Sect. 4.

By $\sum_j$ we always denote a summation taken over all $j$ from $\{0,1,\dots,n\}$. If conditions are stated for the summation, they are applied to this default range $j$ from $\{0,1,\dots,n\}$.

## 2. Results

For the sake of simplicity, we assume that $\mathbf{P}\{\xi_0=0\}=0$. To treat the general case it is enough to study in the same way the behavior of the roots on the sets $\{\alpha_n=k,\beta_n=l\}$, where

$$\alpha_n=\max\{j=0,\dots,n\,:\,\xi_j\neq 0\},\qquad \beta_n=\min\{j=0,\dots,n\,:\,\xi_j\neq 0\}.$$

**Theorem 1.** *If the distribution of $|\xi_0|$ has a slowly varying tail, then for any $\delta > 0$*

$$\mathbf{P}\left\{R_n(e^{-\delta/n}, e^{\delta/n}) = 0\right\} \longrightarrow 1, \quad n \to \infty.$$

Consider the index $\tau = \tau_n \in \{0, \ldots, n\}$ such that $|\xi_\tau| \geqslant |\xi_j|$ for $j = 0, \ldots, n$. If it is not unique, we take the minimum one. Let $\omega_1, \ldots, \omega_n$ be the complex roots of the system of equations

$$z^\tau + \frac{\xi_0}{\xi_\tau} = 0, \quad z^{n-\tau} + \frac{\xi_\tau}{\xi_n} = 0.$$

**Theorem 2.** *If the distribution of $\log(1 + \log(1 + |\xi_0|))$ has a slowly varying tail, then for any $\varepsilon \in (0, 1)$*

$$\mathbf{P}\{F_n(\varepsilon)\} \to 1, \quad n \to \infty,$$

*where $F_n(\varepsilon)$ denotes the event that it is possible to enumerate the roots $z_1, \ldots, z_n$ of $G_n$ in such a way that*

$$|z_k - w_k| < \frac{\varepsilon}{n}|w_k|$$

*for $k = 1, \ldots, n$.*

## 3. Auxiliary lemmas

First we need to formulate and prove some auxiliary results. The following result is due to Pellet.

**Lemma 1.** *Let $g(z) = \sum_j a_j z^j$ be a polynomial of degree $n$. Suppose for some $k = 1, \ldots, n-1$ the associated polynomial*

$$\tilde{g}(z) = \sum_{j \neq k} |a_j| z^j - |a_k| z^k$$

*has exactly two positive roots $R$ and $r$, $R > r$. Then $g$ has exactly $k$ roots inside the circle $\{z \in \mathbb{C} : |z| = r\}$ and $n - k$ roots outside the circle $\{z \in \mathbb{C} : |z| = R\}$.*

*Proof.* See, e.g., [7]. □

The next lemma is due to Ostrowski.

**Lemma 2.** *Let $B$ be a closed region in the complex plane, the boundary of which consists of a finite number of regular arcs; let the functions $f(z), h(z)$ be regular on $B$. Assume that for all values of the real parameter $t$, running in the interval $a \leqslant t \leqslant b$, the function $f(z) + t \cdot h(z)$ is non zero on the boundary of $B$. Then the number of the roots of $f(z) + t \cdot h(z)$ inside $B$ is independent of $t$ for $a \leqslant t \leqslant b$.*

*Proof.* See [6]. □

**Lemma 3.** *Consider a monic polynomial of degree $n$ with complex coefficient $g(z) = \sum_j a_j z^j$ such that $a_n = 1, a_0 \neq 0$. Fix some $k = 1, \ldots, n-1$ and denote by $w_1, \ldots, w_{n-k}$ the roots of the equation $z^{n-k} + a_k = 0$. Put*

$$A_k = \sum_{j \neq k} |a_j|.$$

*If for some $\varepsilon > 0$*

$$A_k \leqslant \left(1 - \frac{\varepsilon}{n}\right) \left(\frac{\varepsilon}{n + \varepsilon}\right)^{n-k} |a_k|^{1/(n-k)}, \tag{2}$$

*then $g$ has exactly $n-k$ roots $z_1,\dots,z_{n-k}$ outside the unit circumference and it is possible to enumerate these roots in such a way that*

$$|z_j-w_j|\leqslant\frac{\varepsilon}{n}|w_j|$$

*for $j=1,\dots,n-k$.*

*Proof.* We will prove a stronger version of the Lemma 3. Namely, we will show that the statement holds for the family of polynomials

$$g_t(z)=z^n+a_kz^k+t\sum_{j\neq k,n}a_jz^j,\quad 0\leqslant t\leqslant 1.$$

In particular,

$$g_0(z)=z^n+a_kz^k,\quad g_1(z)=g(z).$$

Let us use Lemma 1 to estimate absolute values of the roots of $g_t$. Consider the associated polynomial

$$\tilde g_t(z)=z^n-|a_k|z^k+t\sum_{j\neq k,n}|a_j|z^j.$$

We have $\tilde g_t(0),\tilde g_t(\infty)>0$ and it follows from (2) that $\tilde g_t(1)<1$. Also, by Descarte's rule of signs, $\tilde g_t$ has at most 2 positive roots. Therefore $\tilde g_t$ has exactly 2 positive roots $r_t$ and $R_t$ such that

$$0<r_t<1<R_t. \tag{3}$$

Now let us show that

$$\left(1-\frac{\varepsilon}{n}\right)|a_k|^{1/(n-k)}\leqslant R_t\leqslant|a_k|^{1/(n-k)}. \tag{4}$$

Since $\tilde g_t(R_t)=0$, we have

$$R_t^{n-k}+t\sum_{j\neq k,n}|a_j|R_t^{j-k}=|a_k|, \tag{5}$$

which proves the right side of (4).

We prove the left side by contradiction. Suppose, on the contrary, that

$$R_t<\left(1-\frac{\varepsilon}{n}\right)|a_k|^{1/(n-k)}.$$

Then

$$\begin{aligned}R_t^{n-k}+t\sum_{j\neq k,n}|a_j|R_t^{j-k}&<\left(1-\frac{\varepsilon}{n}\right)^{n-k}|a_k|+A_kR_t^{n-k-1}\\&\leqslant\left(1-\frac{\varepsilon}{n}\right)^{n-k}|a_k|+A_k\left(1-\frac{\varepsilon}{n}\right)^{n-k-1}|a_k|^{1-\frac{1}{n-k}}\\&=\left(1-\frac{\varepsilon}{n}\right)^{n-k}|a_k|+\frac{A_k}{|a_k|^{1/(n-k)}}\left(1-\frac{\varepsilon}{n}\right)^{n-k-1}|a_k|.\end{aligned}$$

It follows from (2) that

$$\frac{A_k}{|a_k|^{1/(n-k)}}\leqslant\frac{\varepsilon}{n},$$

therefore,

$$R_t^{n-k}+t\sum_{j\neq k,n}|a_j|R_t^{j-k}<\left(1-\frac{\varepsilon}{n}\right)^{n-k}|a_k|+\frac{\varepsilon}{n}\left(1-\frac{\varepsilon}{n}\right)^{n-k-1}|a_k|$$

$$= \left(1 - \frac{\varepsilon}{n}\right)^{n-k-1} |a_k| \leqslant |a_k|,$$

which contradicts with (5). Thus (4) is proved.

It follows from (3), (4) and the Lemma 1 that $k$ roots of $g_t$ lie inside the circle $\{z \in \mathbb{C} : |z| = 1\}$ and the other $n-k$ – outside the circle $\{z \in \mathbb{C} : |z| = (1-\varepsilon/n)|a_k|^{1/(n-k)}\}$ for all $t \in [0,1]$.

Let $z_0$ be a root of $g_t$ from the second group, i.e.,

$$|z_0| > \left(1 - \frac{\varepsilon}{n}\right) |a_k|^{1/(n-k)}. \tag{6}$$

We have

$$|z_0^n + a_k z_0^k| = t \cdot \Big| \sum_{j \neq k,n} a_j z_0^j \Big| \leqslant A_k z_0^{n-1},$$

which leads to

$$\prod_{j=1}^{n-k} |z_0 - w_j| \leqslant A_k |z_0|^{n-k-1}.$$

This implies that there exists an index $l$ such that

$$|z_0 - w_l| \leqslant \left(\frac{A_k}{|z_0|}\right)^{1/(n-k)} |z_0|.$$

Combining this with (2) and (6) we obtain

$$\begin{aligned} |z_0 - w_l| &< \left(\frac{A_k}{(1-\varepsilon/n)|a_k|^{1/(n-k)}}\right)^{1/(n-k)} |z_0| \\ &\leqslant \frac{\varepsilon}{n+\varepsilon}|z_0| \leqslant \frac{\varepsilon}{n+\varepsilon}|w_l| + \frac{\varepsilon}{n+\varepsilon}|z_0 - w_l|, \end{aligned}$$

which produces

$$|z_0 - w_l| < \frac{\varepsilon}{n}|w_l| = \frac{\varepsilon}{n}|a_k|^{1/(n-k)}.$$

It means that all roots of $g_t$ from the second group belong to $\cup_{m=1}^{n-k} B_m$, where $B_m = \{z \in \mathbb{C} : |z - w_m| < \varepsilon |w_m|/n\}$. Since $\varepsilon/n < \sin[\pi/(n-k)]$, all $B_1, \dots, B_{n-k}$ are disjoint. Therefore $g_t$ does not vanish on the boundary of $B_m$ for all $t \in [0,1], m = 1, \dots, n-k$. To conclude the proof, it remains to show that every $B_m$ contains exactly one root of $g_t$. Obviously, this is true for $t = 0$. Therefore, by Lemma 2, this is also true for all $t \in [0,1]$. □

**Lemma 4.** *Let $\{\eta_j\}_{j=0}^{\infty}$ be non-negative i.i.d. random variables. Put*

$$S_n = \sum_j \eta_j, \quad M_n = \max\{\eta_j\}_{j=0}^n.$$

(a) *The distribution of $\eta_0$ has a slowly varying tail if and only if*

$$\frac{M_n}{S_n} \xrightarrow{P} 1, \quad n \to \infty.$$

(b) *The distributin of $\eta_0$ has an infinite mean if an only if*

$$\frac{S_n - M_n}{n} \xrightarrow{a.s.} \infty, \quad n \to \infty.$$

*Proof.* For (a) see [1], for (b) see [5, Theorem 2.1]. □

**Lemma 5.** *Suppose $a_0, a_1, \ldots, a_n \geqslant 0$ and $\varepsilon > 0$. If for some $k = 1, \ldots, n-1$*

$$\prod_{j \neq k} (1 + a_j)^{2n^2} \leqslant 1 + a_k$$

*and*

$$a_k \geqslant 2(1-\varepsilon)^{-4n^2/(4n-1)} \varepsilon^{-4n^3/(4n-1)} (n+\varepsilon)^{4n^3/(4n-1)}, \tag{7}$$

*then*

$$\sum_{j \neq k} a_j + 1 \leqslant \left(1 - \frac{\varepsilon}{n}\right) \left(\frac{\varepsilon}{n+\varepsilon}\right)^{n-k} a_k^{1/(n-k)}.$$

*Proof.* Since $1 + \sum_{j \neq k} a_j \leqslant \prod_{j \neq k}(1 + a_j)$, it suffices to show that

$$(2a_k)^{1/(2n)^2} \leqslant (1-\varepsilon) \left(\frac{\varepsilon}{n+\varepsilon}\right)^n a_k^{1/n},$$

which is equivalent to (7). □

## 4. Proof of theorems

*Proof of Theorem 1.* By Lemma 4 (a), for any $\delta > 0$ we have $\mathbf{P}\{A_n\} \to 1, n \to \infty$, where

$$A_n = \left\{ |\xi_\tau| > e^\delta \sum_{j \neq \tau} |\xi_j| \right\}.$$

Consider the associated polynomial

$$\tilde{G}(z) = \sum_{j \neq \tau} |\xi_j| z^j - |\xi_\tau| z^\tau.$$

Suppose $A_n$ occurs. If $1 \leqslant t \leqslant e^{\delta/n}$, then

$$|\xi_\tau t^\tau| > e^\delta \sum_{j \neq \tau} |\xi_j| \geqslant t^n \sum_{j \neq \tau} |\xi_j| \geqslant \Big| \sum_{j \neq \tau} t^j \xi_j \Big|.$$

If $e^{-\delta/n} \leqslant t \leqslant 1$, then

$$|\xi_\tau t^\tau| \geqslant e^{-\delta} |\xi_\tau| > \sum_{j \neq \tau} |\xi_j| \geqslant \Big| \sum_{j \neq \tau} t^j \xi_j \Big|.$$

Therefore $\tilde{G}$ does not have real roots in the interval $[e^{-\delta/n}, e^{\delta/n}]$. Further, $\tilde{G}(0) > 0, \tilde{G}(\infty) > 0$, and $\tilde{G}(1) < 0$. By Descarte's rule of signs $\tilde{G}$ has at most 2 positive roots. Thus $\tilde{G}$ has exactly 2 positive roots $r$ and $R$ such that

$$0 < r < e^{-\delta/n} < e^{\delta/n} < R.$$

By Lemma 1, $G$ has exactly $\tau$ roots inside the circle $\{z \in \mathbb{C} : |z| = e^{-\delta/n}\}$ and $n - \tau$ roots outside the circle $\{z \in \mathbb{C} : |z| = e^{\delta/n}\}$. Therefore, $A_n$ implies that $R_n(e^{-\delta/n}, e^{\delta/n}) = 0$ which concludes the proof. □

*Proof of Theorem 2.* Consider the events

$$A_n = \left\{ \prod_{j \neq \tau} \left(1 + \frac{|\xi_j|}{|\xi_n|}\right)^{2n^2} \leqslant 1 + \frac{|\xi_\tau|}{|\xi_n|} \right\}$$

and

$$B_n = \left\{ \frac{|\xi_\tau|}{|\xi_n|} \geqslant 2(1-\varepsilon)^{-4n^2/(4n-1)} \varepsilon^{-4n^3/(4n-1)} (n+\varepsilon)^{4n^3/(4n-1)} \right\}.$$

Since the distribution of $\log(1+\log(1+|\xi_0|))$ has a slowly varying tail, by Lemma 4 (a),

$$\mathbf{P}\left\{ 4 \cdot \sum_{j \neq \tau} \log(1+\log(1+|\xi_j|)) \leqslant \log(1+\log(1+|\xi_\tau|)) \right\} \to 1, \quad n \to \infty,$$

which implies

$$\mathbf{P}\left\{ \left( \sum_{j \neq \tau} \log(1+|\xi_j|) \right)^4 \leqslant \log(1+|\xi_\tau|) \right\} \to 1, \quad n \to \infty. \tag{8}$$

Since $\mathbf{E}\log(1+|\xi_0|) = \infty$, by Lemma 4 (b) with probability one

$$\frac{1}{n} \sum_{j \neq \tau} \log(1+|\xi_j|) \to \infty, \quad n \to \infty,$$

which together with (8) produces

$$\mathbf{P}\left\{ n^3 \cdot \sum_{j \neq \tau} \log(1+|\xi_j|) \leqslant \log(1+|\xi_\tau|) \right\} \to 1, \quad n \to \infty,$$

and

$$\mathbf{P}\left\{ \log(1+|\xi_\tau|) \geqslant n^4 \right\} \to 1, \quad n \to \infty.$$

Since for any $\delta > 0$ there exists $T > 0$ such that $\mathbf{P}\{T^{-1} < |\xi_n| < T\} > 1-\delta$, the last two inequalities imply

$$\mathbf{P}\{A_n\}, \mathbf{P}\{B_n\} \to 1, \quad n \to \infty.$$

By Lemma 5, the event $A_n \cap B_n$ implies that the polynomial $G_n(z)/|\xi_n|$ satisfies the conditions of Lemma 3. Thus we have proved the theorem for the roots of $G_n$ lying outside the unit circumference. To treat the rest of the roots consider the associated polynomial

$$G_n^*(z) = z^n G(1/z) = \sum_j \xi_j z^{n-j}$$

and note that $z_0$ is a root of $G_n$ if and only if $z^{-1}$ is a root of $G_n^*(z)$.

□

## References


[1] D. Darling, *The Influence of the Maximum Term in the Addition of Independent Random Variables*, Trans. Amer. Math. Soc., 73 (1952), pp. 95–107.

[2] J. M. Hammersley, *The zeroes of a random polynomial*, in Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. Vol. II, University California Press, Berkeley, Los Angeles, 1956, pp. 89–111.

[3] I. Ibragimov and O. Zeitouni, *On roots of random polynomials*, Trans. Amer. Math. Soc., 349 (1997), pp. 2427–2441.

[4] I. Ibragimov and D. Zaporozhets, *On distribution of zeros of random polynomials in complex plane*, arXiv:1102.3517v1, 2011 (to appear in Prokhorov Festschrift)

[5] H. Kesten and R. Maller, *The Effect of Trimming on the Strong Law of Large Numbers*, Proc. London Math. Soc., 71 (1995), pp. 441–480.

[6] A. M. Ostrowski, *Solution of equations in Euclidean and Banach spaces*, New York: Academic Press, 1973.
[7] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, New York: Oxford University Press, 2002.
[8] L. Shepp and R.J. Vanderbei, *The complex zeros of random polynomials*, Trans. Amer. Math. Soc., 347 (1995), pp. 4365–4383.
[9] D. I. Shparo and M. G. Shur, *On the distribution of roots of random polynomials*, Vestnik Moskov. Univ. Ser. I Mat. Meh., 3 (1962), pp. 40–43. (in Russian)
[10] D. N. Zaporozhets, *An example of a random polynomial with unusual zeros behavior*, Teor. Ver. Prim., 50 (2005), pp. 549–555. (English translation: Theory Probab. Appl. )